Maximum speed of quantum gate operation

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Abstract

We consider a quantum gate, driven by a general time-dependent Hamiltonian, that complements the state of a qubit and then adds to it an arbitrary phase shift. It is shown that the minimum operation time of the gate is $\tau = \frac{h}{4E}(1 + 2\frac{\theta}{\pi})$, where h is Planck's constant, E is the average over time of the quantum-mechanical average energy, and θ is the phase shift modulo π.

It had been shown in [3] that there exists a fundamental limit to the speed of dynamical evolution of a quantum system. Namely, the minimum time required for a system to go from a given state to one orthogonal to it is

$$\tau = \frac{h}{4E}, \tag{1}$$

where h is Planck's constant and E is the quantum-mechanical average energy of the system. Expression (1) applies to the *autonomous* time evolution of a system, and it is not immediately applicable to changes in the system state caused by the interaction with another (external) system.

This paper considers the question of what is the minimum time of operation of quantum gates that operate on qubits (i.e., quantum systems with two-dimensional Hilbert space).

Let

$$\psi_1(0) = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] \quad \text{and} \quad \psi_2(0) = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] \tag{2}$$

be the two initial orthogonal stationary states of a qubit. Consider a "gate" that complements the state of the qubit (a quantum inverter or a controlled-NOT gate with the controlling qubit in logical state 1) and then adds to it an arbitrary phase shift θ. This is a device that applies an external interaction to the system for a certain time τ such that at the end of this time interval

$$\psi_1(\tau) = \psi_2(0)e^{-i\theta} \quad \text{and} \quad \psi_2(\tau) = \psi_1(0)e^{-i\theta}, \tag{3}$$

i.e., the two orthogonal states are swapped and a given phase shift θ is added to the resulting state. Note that, owing to linearity, (3) is a necessary and sufficient condition for an *unknown* quantum state $a\psi_1(0) + b\psi_2(0)$ of a qubit to be converted into the ortogonal state with a phase shift θ, provided that $\mathrm{Re}(ab^*) = 0$ (this condition specifies a two-parameter family of states). Note also that the overall phase of the state is essential, since this qubit is intended to be part of a many-qubit system.

This problem was solved in [2] for the case of a time-independent Hamiltonian. Here we treat the general case of a *time-dependent* Hamiltonian. As in [3], we assume that the energy of the system is nonnegative (in other words, we define energy relative to the energy of the ground state of the system). Thus, the general form of the Hamiltonian we consider is

$$\mathbf{H}(t) = f(t)\begin{bmatrix} E_{11} & E_{12}e^{i\phi} \\ E_{12}e^{-i\phi} & E_{22} \end{bmatrix} = f(t)\mathbf{H}_0, \tag{4}$$

where $f(t) > 0$ and $\mathbf{H}_0$ is a nonnegative definite self-adjoint operator—which is equivalent to

$$E_{11} \geq 0, \quad E_{22} \geq 0, \quad \text{and} \quad E_{11}E_{22} - E_{12}^2 \geq 0. \tag{5}$$

The time evolution of a system driven by a time-dependent Hamiltonian is usually analyzed within the framework of perturbation theory (e.g., [1]). Here, however, we are interested in an *exact* solution.

Let

$$\psi(t) = a_1(t)\psi_1(0) + a_2(t)\psi_2(0); \tag{6}$$

then the Schrödinger equation for ψ results in the following system of differential equations for $a_1(t)$ and $a_2(t)$:

$$\begin{aligned} i\hbar\frac{da_1(t)}{dt} &= f(t)[E_{22}a_1(t) + E_{12}a_2(t)e^{-i\phi}] \\ i\hbar\frac{da_2(t)}{dt} &= f(t)[E_{12}a_1(t)e^{i\phi} + E_{11}a_2(t)]. \end{aligned} \tag{7}$$

Let us introduce functions $b_1(t)$ and $b_2(t)$,

$$b_i(t) = k_i a_1(t) + a_2(t), \quad i = 1, 2, \tag{8}$$

where

$$\begin{aligned} k_1 &= \frac{e^{i\phi}}{2E_{12}}\left[E_{22} - E_{11} + \sqrt{(E_{11} - E_{22})^2 + 4E_{12}^2}\right] \\ k_2 &= \frac{e^{i\phi}}{2E_{12}}\left[E_{22} - E_{11} - \sqrt{(E_{11} - E_{22})^2 + 4E_{12}^2}\right]. \end{aligned} \tag{9}$$

The equations for the $b_i(t)$ are readily solved, yielding

$$\begin{aligned} b_1(t) &= c_1 e^{-\frac{iE_1}{\hbar}F(t)}, \\ b_2(t) &= c_2 e^{-\frac{iE_2}{\hbar}F(t)}. \end{aligned} \tag{10}$$

Here $F(t) = \int_0^t f(t)dt$; c_1 and c_2 are constants depending on the initial conditions; and

$$\begin{aligned} E_1 &= \frac{1}{2}\left[E_{11} + E_{12} + \sqrt{(E_{11} - E_{22})^2 + 4E_{12}^2}\right], \\ E_2 &= \frac{1}{2}\left[E_{11} + E_{12} - \sqrt{(E_{11} - E_{22})^2 + 4E_{12}^2}\right] \end{aligned} \tag{11}$$

are eigenvalues of $\mathbf{H}_0$.

From (8) we obtain

$$\begin{aligned} a_1(t) &= \frac{1}{k_1 - k_2}[b_1(t) - b_2(t)], \\ a_2(t) &= \frac{1}{k_1 - k_2}[k_1 b_2(t) - k_2 b_1(t)]. \end{aligned} \tag{12}$$

Now, let

$$\psi_i = a_{i1}(t)\psi_1(0) + a_{i2}(t)\psi_2(0), \quad i = 1, 2, \tag{13}$$

the initial conditions being

$$\begin{aligned} a_{11}(0) = 1, &\quad a_{12}(0) = 0, \\ a_{21}(0) = 0, &\quad a_{22}(0) = 1. \end{aligned} \tag{14}$$

It follows from (10) and (14) that

$$\begin{aligned} a_{11}(t) &= \frac{1}{k_1 - k_2}\left[k_1 e^{-\frac{iE_1}{\hbar}F(t)} - k_2 e^{-\frac{iE_2}{\hbar}F(t),}\right] \\ a_{12}(t) &= \frac{k_1 k_2}{k_1 - k_2}\left[e^{-\frac{iE_2}{\hbar}F(t)} - e^{-\frac{iE_1}{\hbar}F(t)}\right], \\ a_{21}(t) &= \frac{1}{k_1 - k_2}\left[e^{-\frac{iE_1}{\hbar}F(t)} - e^{-\frac{iE_2}{\hbar}F(t)}\right], \\ a_{22}(t) &= \frac{1}{k_1 - k_2}\left[k_1 e^{-\frac{iE_2}{\hbar}F(t)} - k_2 e^{-\frac{iE_1}{\hbar}F(t)}\right]. \end{aligned} \tag{15}$$

Conditions (3) require that

$$a_{11}(\tau) = a_{22}(\tau) = 0 \tag{16}$$

and

$$a_{12}(\tau) = a_{21}(\tau) = e^{-i\theta}. \tag{17}$$

It follows from (15) that, in order to satisfy (16), one should have

(19)

$E_1 = E_{11} + E_{12}, \quad E_2 = E_{11} - E_{12}.$

Now, we obtain from (16) that

$$\cos\left(\frac{E_{12}}{\hbar}F(\tau)\right) = 0; \tag{20}$$

then (17) leads to the condition

$$e^{i\phi} = e^{-i\phi}, \quad \text{i.e.,} \quad \phi = n\pi, \quad n = 0, 1, 2, \dots . \tag{21}$$

Also, from (17),

$$\begin{aligned} a_{12}(\tau) = a_{21}(\tau) &= \pm i e^{-\frac{i}{\hbar}E_{11}F(\tau)} \sin\left(\frac{E_{12}}{\hbar}F(\tau)\right) \\ &= e^{-i\theta}, \end{aligned} \tag{22}$$

where the plus sign corresponds to the choice $\phi = \pi$ and the minus to the choice $\phi = 0$.

From (20),

$$\sin\left(\frac{E_{12}}{\hbar}F(\tau)\right) = \pm 1. \tag{23}$$

Taking into account that $F(t)$ is a nonnegative and monotonically increasing function of t, we conclude that, to achieve minimum τ, it must be that

$$\frac{E_{12}}{\hbar}F(\tau) = \frac{\pi}{2} \tag{24}$$

and

$$\frac{E_{11}}{\hbar}F(\tau) = \begin{cases} \text{either} & \theta + \frac{\pi}{2} \\ \text{or} & \theta - \frac{\pi}{2} \end{cases}. \tag{25}$$

Therefore,

$$\text{either} \quad \frac{E_{11}}{E_{12}} = \frac{2\theta}{\pi} + 1 \quad \text{or} \quad \frac{E_{11}}{E_{12}} = \frac{2\theta}{\pi} - 1, \tag{26}$$

the second choice being applicable only if $\theta \geq \pi$.

Now, let us calculate the average over time, E, of the quantum-mechanical average energy of the system:

$$E = \frac{1}{\tau}\int_0^\tau \langle\psi_i(t)|\mathbf{H}(t)|\psi_i(t)\rangle dt = \frac{E_{11}}{\tau}F(t). \tag{27}$$

From (24) and (27) we obtain

$$\tau = \frac{\pi\hbar}{2E}\cdot\frac{E_{11}}{E_{12}} = \frac{h}{4E}\cdot\frac{E_{11}}{E_{12}}. \tag{28}$$

Thus, to obtain the minimum value of τ we must take the smallest ratio E_{11}/E_{12}.

Finally, by (26),

$$\tau = \tau(\theta) = \frac{h}{4E}\left[1 + 2\frac{\theta \bmod \pi}{\pi}\right]. \tag{29}$$

It is remarkable that expression (29) is exactly the same as that obtained in [2] for the case of the time-independent Hamiltonian.

As a numerical example, consider experiments [4] made with Ca^+ ions in an ion trap. The characteristic wavelength of the transition between the two relevant Ca^+ energy levels is $\lambda = 397\,\text{nm}$, which yields $\tau = \frac{\lambda}{2c} \sim 6.62\cdot 10^{-16}\,\text{s}$.

Consider now a one-qubit quantum gate that makes an arbitrary unitary transformation of a *known* state such that the absolute value of the inner product

$$\left|\langle\psi(\tau_\alpha)|\psi(0)\rangle\right| = \cos\alpha. \tag{30}$$

A similar analysis shows that, for any α ($0 \le \alpha \le \frac{\pi}{2}$), the minimum time required for this operation is

$$\tau_\alpha = \frac{\alpha h}{2\pi E} = \frac{2\alpha}{\pi}\tau(0). \tag{31}$$

Expression (30) is, again, exactly the same as in the case of the time-independent Hamiltonian.

It should be pointed out that the speed of quantum gates which have been implemented up to now is very far from the fundamental limit (29). In particular, [5] gives an excellent theoretical analysis, as well as experimental results, of the operation time of quantum gates that operate on trapped-ion qubits using laser pulses that entangle the electronic and vibrational degrees of freedom of the trapped $^{40}Ca^+$ ions. The gate time per ion obtained in [5] is of the order of $10^{-9}\,\text{s}$.

References

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